Derivation of var(T), where $T \sim t_\nu$

$var(T) =$

7.3 The Central Limit Theorem (CLT)

- We know that when $Y_1, \ldots, Y_n$ are sampled from a $N(\mu, \sigma^2)$ population, then $\bar{Y}$ has a normal sampling distribution, i.e.,

- What if our population distribution is not normal?
- The CLT tells us the sampling distribution of $\bar{Y}$ will be approximately normal as long as the sample size is large.
- See simulation examples.

Central Limit Theorem: Let $Y_1, \ldots, Y_n$ be iid r.v.'s with $E(Y_i) = \mu$ and $var(Y_i) = \sigma^2 < \infty$, $i = 1, \ldots, n$. Then the cdf of $\dfrac{\bar{Y} - \mu}{\sigma/\sqrt{n}}$ converges to a

Implications:

- The proof of the CLT uses the following results:

1. Convergence of mgf's implies convergence of cdf's. That is, if $\{Y_n\}$ is a sequence of r.v.'s, if $m_{Y_n}(t) \longrightarrow m_Y(t)$ pointwise in a neighborhood of 0 as $n \to \infty$, then $F_{Y_n}(y) \longrightarrow F_Y(y)$ pointwise as $n \to \infty$. In this case, we say Y_n ____________________ to Y and we write $Y_n \xrightarrow{d} Y$.

2. If a sequence of numbers $\{a_n\} \longrightarrow a$ as $n \to \infty$, then:

Proof of CLT: Let $U_n = \sqrt{n}\left(\frac{\bar{Y}-\mu}{\sigma}\right)$.

We will show $U_n \xrightarrow{d} N(0,1)$. Now $Y_1, \dots, Y_n$ each have mgf $m_Y(t)$. Define $W_i = \frac{Y_i - \mu}{\sigma}$ and let $m_W(t)$ denote the mgf of $W_1, \dots, W_n$.

By Taylor's formula, there exists t^*, $0 < t^* < t$, such that:

- In practice, our sample size n is always finite. How large should n be for the approximation to be "good"?

① The larger n is, the better the normal approximation.

② The more symmetric the data's density $f_Y(y)$ is, the better the approximation is.

- If the population distribution is symmetric, $\bar{Y}$ may have an approximately normal sampling distribution if n is around
- If the distribution is highly skewed, n may need to be around for the distribution of $\bar{Y}$ to be approximately normal.

<u>Example</u>: The service times for customers at a cashier are iid r.v.'s with mean 2.5 minutes and standard deviation 2 minutes. Approximately what is the probability that the cashier will take more than 4 hours to serve 100 people?

7.5 The Normal Approximation to the Binomial

- A common example of using the CLT is in the calculation of binomial probabilities.
- Suppose Y is a binomial r.v. (i.e., Y = # of "successes" in a binomial experiment) with # of trials $= n$ and common success probability $= p$.
- Then if $X_1, \ldots, X_n$ are iid Bernoulli (p) r.v.'s, we see:

- And the sample proportion

- If n is large, then the CLT applies and:

Example 1: A machine will be shut down for repairs if a daily random sample of 100 items reveals at least 15% of the sampled items are defective. Suppose the machine is actually producing 10% defective items that day. What is the probability the machine will be shut down?

Note: Since we are approximating a discrete distribution (binomial) with a continuous distribution (normal), a continuity correction will typically improve the approximation:

Picture:

So:

- How large should n be for the normal approximation to work well?
- If the binomial distribution is <u>not</u> <u>too</u> <u>skewed</u>, then n only needs to be moderately large (around 15 or 20).
- The binomial distribution is more skewed if p is
- <u>Rule of Thumb:</u> If

then the normal approximation works well.

<u>Example 2:</u> An airline knows that generally around 5% of ticket-holders do not show up for their flight. For a certain flight with 155 seats, the airline sells 160 tickets. What is the probability that everyone who shows up will get a seat?

Let Y = # who show up.

Then $Y \sim$

<u>Picture:</u>

- Note: The normal distribution can be used to approximate other discrete distributions, such as the Poisson.

- Exercise 7.100 shows that if $Y \sim \text{Poisson}(\lambda)$, then: