4.9 Moment-generating functions for Continuous r.v.'s

- Moments and the mgf are defined similarly as in the discrete case.

Defn: If Y is a continuous r.v., the k-th moment (about the origin) of Y is:

Example: (Uniform moments): If $Y \sim Unif(0,10)$, find a formula for the k-th moment of Y.

Defn: If Y is a continuous r.v., the mgf of Y is

if $m_Y(t)$ exists, i.e., if $E(e^{tY}) < \infty$ for t in some open neighborhood $(-b, b)$ around 0.

- Again, $E(Y^k) = m_Y^{(k)}(0)$ if $m_Y(t)$ exists.
- The proof of this is identical to the proof in the discrete case, with integrals replacing sums for expected values.

4.5 The Normal Distribution

- The <u>normal distribution</u> is probably the continuous distribution that is most commonly used as a model in statistics.
- Many real-world data sets follow an approximately normal distribution.
- The normal pdf is defined over the entire real line:

<u>Defn:</u> A r.v. Y has a <u>normal</u> distribution [Shorthand: $Y \sim N(\mu, \sigma^2)$] if its pdf is:

where $\sigma > 0$ and $-\infty < \mu < \infty$.

- The normal density is characterized by its "bell" shape:

<u>Properties of the normal pdf:</u>

(i) The pdf is symmetric about μ, i.e., for any real number a:

(ii) $f(y)$ has inflection points located at

(iii) $\lim_{y \to -\infty} f(y) = \lim_{y \to \infty} f(y) =$

<u>Theorem:</u> The normal pdf is a valid density function.

<u>Proof</u>: Clearly $f(y) \geq 0$ for all y.
(In fact, $f(y) > 0$ for all y.)

- We must show:

Let $I =$

Letting $z = \frac{y-\mu}{\sigma}$ so that

Theorem: If $Y \sim N(\mu, \sigma^2)$, then the mgf of Y is:

Proof: $m_Y(t) = E\left[e^{tY}\right] =$

Theorem: (Normal mean and variance):
If $Y \sim N(\mu, \sigma^2)$, then:

Proof:

Finding Normal Probabilities

- If $Y \sim N(\mu, \sigma^2)$, the cdf for Y is:

but this integral does not exist in closed form.

- Furthermore, any probability $P(a \leq Y \leq b)$

cannot be found analytically.

- It can be approximated via numerical integration.
- We use software or tables to find such normal probabilities.

<u>Defn</u>: A normal r.v. with mean $\mu = 0$ and variance $\sigma^2 = 1$ is called a

__________ __________ r.v.

- We typically denote a $N(0,1)$ r.v. by Z.

<u>Theorem</u>: If $Y \sim N(\mu, \sigma^2)$, then the <u>standardized</u> version of Y,

has a $N(0,1)$ distribution.

<u>Proof</u>:

- The <u>standard normal</u> probabilities $P(Z > z)$ can be found in Table 4 for various values of z.
- <u>Note</u>: Table 4 gives <u>upper-tail</u> probabilities for positive z values.

Picture:

- Probabilities for negative z values can be found by symmetry:

Example 1: If $Z \sim N(0,1)$, find:

$P(Z > 1.83) =$

$P(Z < -0.42)$

$P(Z \leq 1.19)$

$P(-1.26 < Z \leq 0.35)$

<u>Note:</u> Using standardization, we can use Table 4 to find probabilities for <u>any</u> normal r.v.

<u>Example 2:</u> A graduating class has GPAs that follow a normal distribution with mean 2.70 and variance 0.16.

- What is the probability a randomly chosen student has a GPA greater than 2.50?

- What is the probability that a random GPA is between 3.00 and 3.50?

- Exactly 5% of students have GPA above what number?

Example 3: Y is a normal r.v. with $\sigma = 10$. Find μ such that $P(Y < 70) = 0.75$.