Question 1

(a) Let A denote the event that both balls selected are white, and let B_i denote the event that bowl i was selected.

$$P(A) = \sum_{i=1}^{5} P(A|B_i)\,P(B_i)$$

$$= \frac{0}{\binom{5}{2}} \times \frac{1}{5} + \frac{\binom{2}{2}}{\binom{5}{2}} \times \frac{1}{5} + \frac{\binom{3}{2}}{\binom{5}{2}} \times \frac{1}{5} + \frac{\binom{4}{2}}{\binom{5}{2}} \times \frac{1}{5} + \frac{\binom{5}{2}}{\binom{5}{2}} \times \frac{1}{5}$$

$$= \frac{(0+1+3+6+10)}{10 \times 5}$$

$$= \frac{2}{5}$$

(b). $$P(B_3|A) = \frac{P(B_3 \cap A)}{P(A)} = \frac{P(A|B_3)P(B_3)}{P(A)} = \frac{\frac{\binom{3}{2}}{\binom{5}{2}} \times \frac{1}{5}}{\frac{2}{5}} = \frac{3}{20}$$

Question 2 : Page 60—62 in notes.

Question 3 :

$$P(Y=\text{even}) = \sum_{y \in \{2k\}} P(Y=y) = \sum_{k=1}^{\infty} pq^{2k-1} = p \cdot \frac{q}{1-q^2} = \frac{q}{1+q}$$

$$P(X=x) = P(Y-1=x) = P(Y=x+1) = p \cdot q^{x+1-1} = pq^x, \quad x=0,1,2,\cdots$$

$$E(X) = E(Y-1) = EY-1 = \frac{1}{p} - 1 = \frac{q}{p}$$

$$Var(X) = Var(Y-1) = Var(Y) = \frac{q}{p^2}$$

Question 4 :

(a). $$P = \frac{\binom{13}{5}}{\binom{52}{5}} = \frac{1287}{2598960} = 4.95 \times 10^{-4}$$

(b). $$P = \frac{\binom{4}{2}\binom{4}{2}\binom{4}{1}}{\binom{52}{5}} = \frac{144}{2598960} = 5.54 \times 10^{-5}$$

(c) A = 4 aces, B = at least 3 aces

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)} = \frac{\binom{4}{4}\binom{48}{1}}{\binom{4}{3}\binom{48}{2} + \binom{4}{4}\binom{48}{1}} = \frac{48}{4 \times \frac{48 \times 47}{2} + 48} = \frac{1}{95}$$

Question 5

Maclaurin Serial: $-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots \quad , |x|<1$

$$\sum_{y=1}^{\infty} P(Y=y) = \sum_{y=1}^{\infty} \frac{-(1-p)^y}{y \cdot \ln p} = \frac{-1}{\ln p} \sum_{y=1}^{\infty} \frac{q^y}{y} = \frac{-1}{\ln(1-q)} \times -\ln(1-q) = 1$$

$$M_Y(t) = E(e^{tY}) = \sum_{y=1}^{\infty} e^{ty} \cdot P(Y=y) = \sum_{y=1}^{\infty} e^{ty} \cdot \frac{-(1-p)^y}{y \cdot \ln p} = \frac{-1}{\ln p} \sum_{y=1}^{\infty} \frac{[e^t(1-p)]^y}{y}$$

$$= \frac{-1}{\ln p} \times -\ln(1-qe^t) = \frac{\ln(1-qe^t)}{\ln p}$$

$$M_Y'(t) = \frac{1}{1-qe^t} \cdot (-q)e^t \cdot \frac{1}{\ln p} = \frac{-q}{\ln p} \cdot \frac{e^t}{1-qe^t}$$

$$E(Y) = M_Y'(0) = \frac{-q}{p \ln p}$$

$$M_Y''(t) = \frac{-q}{\ln p} \times \frac{e^t(1-qe^t) - e^t(-q)e^t}{(1-qe^t)^2} = \frac{-qe^t}{\ln p \cdot (1-qe^t)^2}$$

$$E(Y^2) = M_Y''(0) = \frac{-q}{(1-q)^2 \ln p}$$

$$Var(Y) = E(Y^2) - E^2(Y) = \frac{-q}{p^2 \ln p} - \frac{q^2}{p^2 \ln^2 p} = \frac{-q(q+\ln p)}{p^2 \ln^2 p}$$